42. **(20 pts)** Use Fermat’s Principle to prove the law of reflection.

While the light could take any path to get from A to B, Fermat’s Principle says it will take the path of least time. We can therefore calculate the time for a path as a function of the reflection point, and then take a derivative to minimize that time.

Assume point A is at height \(a\) above the plane, and point B is at height \(b\) above the plane. Assume that the point A is above the origin along the plane surface, that the reflection occurs at \(x\) on that axis, and that the point B is a distance \(c\) from point A on that axis.

The time for the path will be:

\[
\Delta t = \frac{\sqrt{a^2 + x^2}}{c} + \frac{\sqrt{b^2 + (c-x)^2}}{c}
\]

Set the derivative of this time with respect to position equal to zero:

\[
\frac{d\Delta t}{dx} = \frac{2x}{2c\sqrt{a^2 + x^2}} - \frac{2(c-x)}{2c\sqrt{b^2 + (c-x)^2}} = 0
\]

\[
x \cdot \frac{1}{c \sqrt{a^2 + x^2}} = \frac{(c-x)}{c \sqrt{b^2 + (c-x)^2}}
\]

\[
\frac{\sin(\theta)}{c} = \frac{\sin(\theta_r)}{c}
\]

\[
\theta_i = \theta_r
\]
43. (20 pts) Show that in Young’s double slit experiment, if the top slit is covered with a dielectric film of thickness $t$ and index of refraction $n$, the entire interference pattern is shifted upward by:

$$\theta_{\text{shift}} \approx \frac{(n-1)t}{a}$$

where $a$ is the separation between the slits. Assume small angles, assume the film is thin compared to the slit to screen distance, and don’t consider reflections in the dielectric film.

**SOLUTION**

For small angles we can assume that the bottom beam travels an extra distance $asin(\theta)$, and the top beam goes through a path of thickness $t$ (with index of refraction equal to $n$ rather than 1 for vacuum or air). Each of these will affect the phase lag of the bottom beam relative to the top. The total phase lag is:

$$\delta = \frac{2\pi}{\lambda_o}(\Delta_p + \Delta_r)$$

But we are not considering any reflections here, so $\Delta_r = 0$. The extra path $asin(\theta)$ causes the lower beam to lag in phase. However, the top beam sees a longer optical path than the bottom beam by $nt - t$, causing the bottom beam to lead in phase:

$$\delta = \frac{2\pi}{\lambda_o} \Delta_p = \frac{2\pi}{\lambda_o} (a \sin \theta - (nt - t))$$
Since the irradiance pattern goes as:

\[ I = I_o \cos^2 \left( \frac{\delta}{2} \right) \]

The interference maxima occur at:

\[ \frac{\delta}{2} = m\pi \]

where \( m \) is any integer. This creates the condition:

\[
\frac{2\pi}{2\lambda_o} (a \sin \theta_{\text{max}} - (nt - t)) = m\pi
\]

\[
a \sin \theta_{\text{max}} = m\lambda_o + (nt - t)
\]

\[
\sin \theta_{\text{max}} = \frac{m\lambda_o + t(n-1)}{a}
\]

For small angles:

\[
\theta_{\text{max}} = \frac{m\lambda_o}{a} + \frac{t(n-1)}{a}
\]

The maxima, and therefore the entire pattern, shift up (higher angles) due to the film.
44. (25 pts) A point source of light S is emitting a single wavelength $\lambda_o$ and is situated a small distance $d$ above a plane mirror. A screen stands normal to the mirror at a distance $L$ from S with $L >> d$. Find the intensity of light on the screen as a function of height $y$ above the mirror.

The geometrical interpretation of this problem is much easier if you realize that the light which reflects of the mirror to the screen creates a virtual image of the point source a distance $d$ below the mirror. If an optical axis is placed along the mirror, this is like a Young’s double slit problem with separation $2d$. However, there is a catch in the phase lag. The reflected rays will show a $\pi$ phase shift which must be considered:

$$\delta = \frac{2\pi}{\lambda_o} (\Delta_p + \Delta_r)$$

$$\delta = \frac{2\pi}{\lambda_o} \left( 2d \sin \theta + \frac{\lambda_o}{2} \right)$$

For small angles, $\sin(\theta)$ can be replaced by $y/L$, where $y$ is the height above the mirror:

$$\delta = \frac{4\pi dy}{\lambda_o L} + \pi$$

This phase condition can be put in the expression for irradiance:
\[ I = I_0 \cos^2\left(\frac{\delta}{2}\right) \]

\[ I = I_0 \cos^2 \left( \frac{2\pi dy}{\lambda_o L} + \frac{\pi}{2} \right) \]

*Note that the π/2 changes the cosines to sines:*

\[ I = I_0 \sin^2 \left( \frac{2\pi dy}{\lambda_o L} \right) \]

**Grader:** 6 points for the path contribution to the phase difference, 6 pts for the reflection contribution, 6 pts for correctly switching from \( \theta \) to \( y \), and 7 points for plugging into expression for \( I \) correctly (switch from \( \cos^2 \) to \( \sin^2 \) is not required).

**45. (15 pts).** Design a thin dielectric film to minimize reflection for white light at normal incidence on glass \((n = 1.515)\) in water \((n = 1.33)\). Determine values for the optimum film thickness and index of refraction. Bonus +3: recommend a dielectric for this purpose.

**Solution**

To create an efficient antireflection coating from a thin film, we need (1) destructive interference between the first and second reflections and (2) each reflection to have the same amplitude.
The second condition can be found from the Fresnel’s Equation for reflection at normal incidence:

\[ r = \frac{1 - n}{1 + n} \]

Where \( n \) is the relative index \( n_2/n_1 \) for a beam moving from medium 1 to medium 2. Since the reflection from a dielectric is not large, we can find \( n \) by simply equating \( r \) for each interface:

\[ r_{w-f} = r_{f-g} \]

\[ 1 - \frac{n_f}{n_w} = 1 - \frac{n_g}{n_f} \]

\[ \frac{1 + \frac{n_f}{n_w}}{1 + \frac{n_g}{n_f}} = \frac{n_f}{n_g} \]

Simple expansion of this expression leads directly to:

\[ n_f = \sqrt{n_g n_w} = 1.42 \]

The first condition requires:

\[ 2n_f t = \frac{\lambda_o}{2} \]

The effect of reflections on the phase difference can be ignored since each beam makes one reflection at a higher index medium. Since this coating is for the visible, it should be designed for a wavelength in the middle of the visible spectrum, such as \( \lambda_o = 550 \text{ nm} \). The required film thickness is then:

\[ t = \frac{\lambda_o}{4n_f} = 97 \text{ nm} \]

Cryolite!
46. (20 pts) This is an image of a soap film taken by reflected white light. Explain:
   a. why it is colorful.
   b. why some regions are black (there is a stable soap film there, it is not rupturing).

a) It is colorful because the thickness of the soap film varies across the surface. From our discussion of thin film interference, we know that the reflected irradiance is:

\[ I_p = 0.16I_0 \cos^2 \left( \frac{\delta}{2} \right) \]

So the reflection maxima will depend on the phase factor \( \delta \), and \( \delta \) depends on the wavelength:
\[ \delta = \frac{\Delta_p + \Delta_r}{\lambda_o} \cdot 2\pi \]

b) Black regions have a soap film, but no reflection. This is because the film is very thin (t approaches zero in the equation below). In these regions, the optical path difference will be negligible, but the phase difference will still be \( \pi \), since the external reflection from the top of the film undergoes a \( \pi \) phase shift, and the internal reflection from the back of the film does not.

\[ \delta = \left(2m + \frac{1}{2} \lambda_o\right) \cdot \frac{2\pi}{\lambda_o} \]

47. (15 pts) A thin film of MgF\(_2\) (\( n = 1.38 \)) is deposited onto glass so that its antireflection wavelength is 580 nm under normal incidence. What wavelength is minimally reflected when the light is incident instead at 45 degrees?

Under normal incidence the first and second reflections will cancel if the total path difference is half a wavelength:

\[ \Delta_p + \Delta_r = \frac{1}{2} \lambda_o \]

For two external reflections, the net difference due to reflections is zero. The path difference is twice the thickness times the index of the film:

\[ 2m_f + 0 = \frac{1}{2} \lambda_o \]

So for normal incidence at 580 nm, the film thickness is:

\[ t = \frac{\lambda_o}{4n_f} = \frac{580\text{nm}}{4(1.38)} = 105\text{nm} \]

If the light is incident at 45 degrees, that alters the path length by a factor of \( \cos(\theta_i) \) (but the net phase due to reflection is still zero)

\[ 2m_f \cos \theta_i + 0 = \frac{1}{2} \lambda_o \]

Use Snell’s law to get \( \theta_i = 30.8 \text{ degrees} \). The wavelength that satisfies this condition is:

\[ \lambda_o = 4m_f \cos \theta_i = 498\text{nm} \]