EM spherical waves

\[ \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{...in any coord. system} \]

Differentials depend on coord system:

\[ dl = dr \hat{r} + rd\theta \hat{\theta} + r \sin \theta \, d\phi \hat{\phi} \]

\[ d\tau = r^2 \sin(\theta) \, dr \, d\theta \, d\phi \]

\[ \nabla_s = \frac{\partial s}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial s}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial s}{\partial \phi} \hat{\phi} \]
\[ \nabla \times \nabla \times \vec{V} = \nabla (\nabla \cdot \vec{V}) - \frac{\partial^2 V_x}{\partial x^2} - \frac{\partial^2 V_y}{\partial y^2} - \frac{\partial^2 V_z}{\partial z^2} \]
The vector spherical Laplacian:

\[
\nabla^2 \vec{E} = \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}
\]

\[
\nabla^2 \psi = \mu_o \varepsilon_o \frac{\partial^2 \psi}{\partial t^2}
\]

\(\psi\) : Our scalar wave which represents the field strength.

The spherical Laplacian:

\[
\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}
\]

(This is a good time for a scalar wave equation!)
Unless you are a masochist ..... put the spherical source at the origin!!!!

\[ \nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) \]

\[ = \frac{1}{r^2} \left( 2r \frac{\partial \psi}{\partial r} + r^2 \frac{\partial^2 \psi}{\partial r^2} \right) \]

\[ = \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \]

\[ = \frac{1}{r} \frac{\partial^2 (r \psi)}{\partial r^2} \]
Put back into wave equation:

\[
\frac{1}{r} \frac{\partial^2 (r \psi)}{\partial r^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}
\]

\[
\frac{\partial^2 (r \psi)}{\partial r^2} = \frac{1}{c^2} \frac{\partial^2 (r \psi)}{\partial t^2}
\]

Wave equation for \(r \psi\)

Guess a solution:

\[
r \psi(r,t) = \psi_0 e^{i(k \cdot r - \omega t)}
\]

Note: \(\psi\) (the field strength) and \(\psi_0\) (the field amplitude) don't have the same unit.

\[
\psi(r,t) = \frac{\psi_0}{r} e^{i(k \cdot r - \omega t)}
\]

As \(r\) increases, the field strength decreases
For: \( r \gg a \), spherical waves can be approximated as a plane waves.
Geometrical Optics: light is a “ray” that travels in straight lines

(the “ray” is essentially the wave vector of a plane wave)
Huygens' and Fermat's Principles

All of Geometrical optics boils down to...

Law of Reflection:
\[ \theta_i = \theta_r. \]

Snell's Law:
\[ \frac{\sin(\theta_i)}{\sin(\theta_t)} = \frac{n_2}{n_1}. \]
Huygens' Principle

Every point on a wavefront may be regarded as a secondary source of wavelets.

Planar wavefront:

\[ c \Delta t \]
Huygens': Point source through an aperture:

Ignore the peripheral and back propagating parts!

In geometrical optics, this region should be dark.
Huygens' Reflection

$$\cos(90 - \theta_i) = \frac{c \Delta t}{L}$$

$$\cos(90 - \theta_r) = \frac{c \Delta t}{L}$$

$L$
Huygens': Refraction

\[
\sin(\theta) = \frac{n - \nu}{\nu} \sin(\theta')
\]

\[
\frac{\sin(\theta)}{\nu} = \frac{x}{\nu \Delta t}
\]

\[
\frac{\sin(\theta)}{\nu} = \frac{x}{\nu \Delta t}
\]
Fermat's Principle

The path a beam of light takes between two points is the one which is traversed in the least time.

Isotropic medium: constant velocity.

Minimum time = minimum path length.
Fermat's: Refraction

\[ t = \frac{AO}{v_i} + \frac{OB}{v_t} \]

\[ t = \frac{\sqrt{a^2 + x^2}}{v_i} + \frac{\sqrt{b^2 + (c-x)^2}}{v_t} \]

\[ \frac{dt}{dx} = \frac{x}{v_i \sqrt{a^2 + x^2}} - \frac{c-x}{v_t \sqrt{b^2 + (c-x)^2}} \]

\[ \frac{dt}{dx} = \frac{\sin(\theta_i)}{v_i} - \frac{\sin(\theta_t)}{v_t} = 0 \]

\[ n_i \sin(\theta_i) = n_t \sin(\theta_t) \]
Huygens’ Principle and Fermat’s principle provide a qualitative (and quantitative) proof of the law of reflection and refraction within the limit of geometrical optics.